ON THE JACOBSON'S LEMMA

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This note is based on the joint work with Bojan Kuzma, [1].

1. Introduction

Let $\mathcal{X}$ be a finite dimensional complex vector space. By $L(\mathcal{X})$ we denote the algebra of all linear operators on $\mathcal{X}$.

For $A,\ B \in L(\mathcal{X})$, let $[A,B] = AB - BA$. The Jacobson's Lemma [3] says that $[A,B]$ is a nilpotent operator if A and $[A,B]$ commute, i.e., if $[A,[A,B]] = 0$. The original proof [3, Lemma 2], and its extension [4], bound the nilindex of $[A,B]$ above by $2^n - 1$, where n is the degree of the minimal polynomial of A. Arguments run as follows.

Denote by $\delta_B := [\,\cdot\,, B]$ the inner derivation on $L(\mathcal{X})$ that is induced by B and rewrite the condition $[A,[A,B]] = 0$ as $\delta_B(A)A = A\delta_B(A)$. It follows that

$$\delta_B(A^2) = \delta_B(A)A + A\delta_B(A) = 2A\delta_B(A)$$

and consequently, by the induction,

$$\delta_B(A^k) = kA^{k-1}\delta_B(A) \qquad (k \in \mathbb{N}).$$

Thus, $\delta_B\big(p(A)\big) = p'(A)\delta_B(A)$, for any polynomial p.

Let $q_A(z)$ be the minimal polynomial of A and let n be its degree. Since $q_A(A) = 0$ we have $q_A'(A)\delta_B(A) = 0$, which is the case $k = 1$ of

$$q_A^{(k)}(A)\delta_B(A)^{2^k-1} = 0. \tag{1}$$

Assume that (1) holds for some $k \in \mathbb{N}$. Then

$$\begin{aligned}
0 &= \delta_B\big(q_A^{(k)}(A)\delta_B(A)^{2^k-1}\big)\\
&= q_A^{(k+1)}(A)\delta_B(A)\delta_B(A)^{2^k-1} + q_A^{(k)}(A)\delta_B\big(\delta_B(A)^{2^k-1}\big)\\
&= q_A^{(k+1)}(A)\delta_B(A)^{2^k} + q_A^{(k)}(A)\delta_B^2(A)\delta_B(A)^{2^k-2} + \\
&\qquad + q_A^{(k)}(A)\delta_B(A)\delta_B^2(A)\delta_B(A)^{2^k-3} + \cdots + q_A^{(k)}(A)\delta_B(A)^{2^k-2}\delta_B^2(A).
\end{aligned}$$

Now, we premultiply this equality with $\delta_B(A)^{2^k-1}$ to get the induction step. Thus, $q_A^{(k)}(A)\delta_B(A)^{2^k-1} = 0$ holds for any $k \in \mathbb{N}$. In particular, for $k = n$, we have

$$n!\delta_B(A)^{2^n-1} = 0,$$

which means that $[A,B]$ is nilpotent with nilindex at most $2^n - 1$. We remark that if $\delta_B^2(A)$ commutes with $\delta_B(A)$, similar arguments would bound nilindex above by $2n - 1$.

In this note we shall improve the estimate on the upper bound of the nilindex of $[A, B]$.

2. Results

Let $q_A(z) = (z - \lambda_1)^{n_1} \cdots (z - \lambda_t)^{n_t}$ be the minimal polynomial of $A \in L(\mathcal{X})$. For each $i \in \{1, \dots, t\}$, let $M_i = \ker (A - \lambda_i)^{n_i}$. Note that $x \in M_i$ if and only if $x \in \ker (A - \lambda_i)^{n_i+\ell}$ for some $\ell \geq 0$.

Lemma 2.1. *If $\delta_A^k(B) M_i = \{0\}$, for some $B \in L(X)$ and $k \in \mathbb{N}$, then M_i is invariant for B.*

Proof. Since $\delta_A = \delta_{A-\lambda}$, for any $\lambda \in \mathbb{C}$, there is no loss of generality if we assume that $\lambda_1 = 0$ and $i = 1$.

Let $x \in M_1$, i.e. $A^{n_1}x = 0$. Clearly, vectors $A^\ell x$ are in M_1, for each $\ell \in \{0, 1, \dots, n_1 - 1\}$. By the assumption, $\delta_A^k(B) A^\ell x = 0$. Thus, with $\ell = n_1 - 1$, we have

$$0 = \delta_A^k(B) A^{n_1-1}x = \sum_{j=0}^{k} \binom{k}{j} A^{k-j} B A^{n_1-1+j} x = A^k B A^{n_1-1} x. \tag{2}$$

The equality $\delta_A^k(B) A^{n_1-2}x = 0$ similarly gives

$$A^k B A^{n_1-2} x - k A^{k-1} B A^{n_1-1} x = 0.$$

If we multiply this last equality by A and use (2), we get $A^{k+1} B A^{n_1-2} x = 0$. Using induction backwards we are left with $A^{k+n_1} B x = 0$, which means that $Bx \in M_1$. □

Lemma 2.2. *If $A \in L(\mathcal{X})$ is a nilpotent operator with nilindex $n \geq 1$, then the inner derivation δ_A is a nilpotent operator on $L(\mathcal{X})$ with nilindex $2n-1$.*

Proof. For a nilpotent operator A with nilindex n, we have

$$\delta_A^{2n-1}(T) = \sum_{j=0}^{2n-1} (-1)^j \binom{2n-1}{j} A^{2n-1-j} T A^j = 0 \qquad (T \in L(\mathcal{X})),$$

which shows that $(\delta_A)^{2n-1} = 0$.

On the other hand, let $x \in \mathcal{X}$ and $T \in L(\mathcal{X})$ be such that $A^{n-1}x \neq 0$ and $TA^{n-1}x = x$. Then

$$\begin{aligned} \delta_A^{2n-2}(T)x &= \sum_{j=0}^{2n-2} (-1)^j \binom{2n-2}{j} A^{2n-2-j} T A^j x \\ &= (-1)^{n-1} \binom{2n-2}{n-1} A^{n-1} T A^{n-1} x = (-1)^{n-1} \binom{2n-2}{n-1} x \neq 0 \end{aligned}$$

gives $(\delta_A)^{2n-2} \neq 0$. □

Lemma 2.3. *Let $A \in L(\mathcal{X})$ be a nilpotent operator with nilindex $n \geq 1$ and let $B \in L(\mathcal{X})$ be such that $\delta_A^2(B) = 0$. Then $\big(\delta_A(B)\big)^{2n-1} = 0$.*

Proof. The classical proof of The Kleinecke-Shirokov Theorem [2, Solution 184] shows that $\delta_A^2(B) = 0$ implies

$$\delta_A^{2n-1}\big(B^{2n-1}\big) = (2n-1)!\big(\delta_A(B)\big)^{2n-1}.$$

By Lemma 2.2, δ_A is a nilpotent operator with nilindex $2n-1$. Thus, $\big(\delta_A(B)\big)^{2n-1} = 0$. □

Theorem 2.4. *Let $q_A(z) = (z-\lambda_1)^{n_1}\cdots(z-\lambda_t)^{n_t}$ be the minimal polynomial of $A \in L(\mathcal{X})$. If $[A,[A,B]] = 0$ for some $B \in L(\mathcal{X})$, then $[A,B]$ is a nilpotent operator with nilindex at most $m := 2\cdot\max\{n_1,\ldots,n_t\} - 1$.*

Proof. For each $1 \le i \le t$, let $M_i := \ker(A-\lambda_i)^{n_i}$. Then $\mathcal{X} = M_1 \oplus\cdots\oplus M_t$. Since $[A-\lambda_i,[A-\lambda_i,B]] = [A,[A,B]] = 0$ we have $[A-\lambda_i,[A-\lambda_i,B]]M_i = \{0\}$. By Lemma 2.1, M_i is invariant for B, which means that we may consider the restrictions to M_i of the involved operators. The restriction of $A-\lambda_i$ to M_i is a nilpotent operator with nilindex n_i. It follows that the local nilindex of $[A-\lambda_i,B] = [A,B]$ on M_i is at most $2n_i - 1$, by Lemma 2.3. Let $x = x_1 \oplus\cdots\oplus x_t$ be the decomposition of $x \in \mathcal{X}$ with $x_i \in M_i$. Then, of course, $[A,B]^{2m-1}x = 0$, where $m = \max\{n_1,\ldots,n_t\}$. □

Corollary 2.5. *If A is a diagonalizable matrix then $[A,[A,B]] = 0$ implies $[A,B] = 0$.*

Proof. The minimal polynomial of A is a product of distinct linear factors. □

References

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