HILDEBRANDT'S THEOREM FOR THE ESSENTIAL SPECTRUM

JANKO BRAČIĆ AND CRISTINA DIogo

Abstract. We prove a variant of the Hildebrandt's Theorem which asserts that the convex hull of the essential spectrum of an operator $A$ on a complex Hilbert space is equal to the intersection of the essential numerical ranges of operators which are similar to $A$. As a consequence, it is given a necessary and sufficient condition for zero not being in the convex hull of the essential spectrum of $A$.

The notion of the numerical range of an element in a normed algebra is well known and extensively studied during last five decades (see the standard references [2, 3, 5, 6]). In this note we confine our self to unital $C^*$-algebras.

If $A$ is a $C^*$-algebra with the identity $1$, then let $A^*$ denote its topological dual and let $\mathcal{P} = \{ f \in A^*; \quad f(1) = 1 = \|f\| \}$ be the set of all normalized states on $A$. The numerical range of an element $a \in A$ is defined by

\begin{equation}
V(a) = \{ f(a); \quad f \in \mathcal{P} \}.
\end{equation}

This set is compact, convex and contains the spectrum $\sigma(a)$ (see [9, Theorem 1]). If $A$ is the $C^*$-algebra $B(\mathcal{H})$ of all bounded linear operators on a complex Hilbert space $\mathcal{H}$, then ([9, Corollary on p. 420])

\begin{equation}
V(T) = W(T) \quad \text{for any} \quad T \in B(\mathcal{H}),
\end{equation}

where

\begin{equation}
W(T) = \{ \langle Tx, x \rangle; \quad x \in \mathcal{H}, \|x\| = 1 \}
\end{equation}

is the usual numerical range.

As already mentioned, the spectrum of $a \in A$ is contained in the numerical range of $a$. Since the numerical range is convex one has

\begin{equation}
\text{conv}(\sigma(a)) = V(a).
\end{equation}

Denote by $\text{inv}(A)$ the set of all invertible elements in $A$. In the case of the $C^*$-algebra $B(\mathcal{H})$, Hildebrandt has proved the following result (see [7]).

2010 Mathematics Subject Classification. 47A10, 47A12.

Key words and phrases. Essential spectrum, essential numerical range, Hildebrandt's theorem.

The first author was supported by the Slovenian Research Agency through the research program P1-0222 and the second author was partially supported by FCT/Portugal through projects PTDC/MAT/121837/2010 and PEst-OE/EEI/LA0009/2013.
Theorem 1 (Hildebrandt’s Theorem). For every \( A \in \mathcal{B}(\mathcal{H}) \),
\[
\overline{\mathrm{conv}}(\sigma(A)) = \bigcap_{s \in \text{inv}(\mathcal{A})} W(s\sigma^{-1}).
\]

We include here a slightly more general version of the Hildebrandt’s theorem. The proof relies on the following lemma by Murphy and West [8]. For the sake of completeness we include its proof. We denote by \( r(a) \) the spectral radius of \( a \in \mathcal{A} \).

Lemma 2. Let \( \mathcal{A} \) be a \( C^* \)-algebra and \( a \in \mathcal{A} \). For any \( \varepsilon > 0 \), there exists \( s \in \text{inv}(\mathcal{A}) \) such that \( \| s^{a}\| < r(a) + \varepsilon \).

Proof. Let \( b = \frac{1}{r(a) + \varepsilon} a \). Then \( r(b) < 1 \), i.e., by the Gelfand-Beurling formula, \( \lim_{n \to \infty} \| b^n \|^{1/n} < 1 \). It follows that the series \( \sum_{n=0}^{\infty} \| (b^n)b^n \| = \sum_{n=0}^{\infty} \| b^n \|^2 \) converges. Hence \( c = \sum_{n=0}^{\infty} (b^n)b^n \in \mathcal{A} \) and \( c \geq 1 \). Let \( s = \sqrt{c} \). Then \( s \geq 1 \), which means that it is invertible. Since \( 0 \leq 1 - s^{-2} \leq 1 \), we have
\[
\| s^{a\varepsilon} \|^2 = \| s^{-1}(s^2 - 1)\|^2 = \| 1 - s^{-2} \| = r(1 - s^{-2}) < 1.
\]
It is obvious now that \( \| s^{a\varepsilon} \| < r(a) + \varepsilon \). \(\square\)

Theorem 3. Let \( \mathcal{A} \) be a \( C^* \)-algebra and \( a \in \mathcal{A} \). Then
\[
\overline{\mathrm{conv}}(\sigma(a)) = \bigcap_{s \in \text{inv}(\mathcal{A})} V(s\sigma^{-1}).
\]

Proof. Let \( a \in \mathcal{A} \). Since \( \sigma(a) \subseteq V(a) \) and, because of the convexity of the numerical range, one actually has \( \overline{\mathrm{conv}}(\sigma(a)) \subseteq V(a) \). Since the spectrum is preserved by similarities one has \( \overline{\mathrm{conv}}(\sigma(a)) \subseteq \bigcap_{s \in \text{inv}(\mathcal{A})} V(s\sigma^{-1}) \).

To prove the other inclusion, let \( \lambda \in \mathbb{C} \setminus \overline{\mathrm{conv}}(\sigma(a)) \). Since \( \overline{\mathrm{conv}}(\sigma(a)) \) is a compact convex set there exists a disk \( \mathbb{D}(\mu, \rho) \) such that \( \lambda \notin \mathbb{D}(\mu, \rho) \) and \( \overline{\mathrm{conv}}(\sigma(a)) \subseteq \mathbb{D}(\mu, \rho) \). Hence \( \lambda - \mu \notin \mathbb{D}(0, \rho) \) and \( \overline{\mathrm{conv}}(\sigma(a) - \mu) \subseteq \mathbb{D}(0, \rho) \), which means that \( r(a - \mu) < \rho \). Let \( \varepsilon > 0 \) be such that \( r(a - \mu) + \varepsilon < \rho \). By Lemma 2, there exists an invertible element \( s \) such that \( \| s(a - \mu)\| < r(a - \mu) + \varepsilon < \rho \). It follows that \( \lambda - \mu \notin W(s(a - \mu)s^{-1}) \) and consequently \( \lambda \notin W(s\sigma^{-1}) \). \(\square\)

Let \( \mathcal{K}(\mathcal{H}) \) be the ideal of all compact operators on a complex Hilbert space \( \mathcal{H} \) and \( \mathcal{C}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \) be the Calkin algebra. For \( A \in \mathcal{B}(\mathcal{H}) \), let \( [A] \) denote the equivalence class \( A + \mathcal{K}(\mathcal{H}) \). If \( [A] \in \text{inv}(\mathcal{C}(\mathcal{H})) \), then \( A \) is said to be a Fredholm operator (see [4, Definition 5.14]). The set of all Fredholm operators in \( \mathcal{B}(\mathcal{H}) \) is denoted by \( \Phi(\mathcal{H}) \).

The essential spectrum of \( A \in \mathcal{B}(\mathcal{H}) \) is defined by
\[
\sigma_{\text{ess}}(A) = \{ \lambda \in \mathbb{C} ; \quad A - \lambda I \notin \Phi(\mathcal{H}) \},
\]
i.e., $\sigma_{\text{ess}}(A) = \sigma([A])$, which means that $\lambda \in \mathbb{C}$ is in the essential spectrum of $A$ if and only if the element $[A - \lambda I]$ is not invertible in the Calkin algebra (see [1]). The essential spectrum $\sigma_{\text{ess}}(A)$ is a non-empty compact subset of $\sigma(A)$. The essential numerical range $W_{\text{ess}}(A)$ of $A \in \mathcal{B}(\mathcal{H})$ is defined analogously, i.e., $W_{\text{ess}}(A)$ is equal to $V([A])$, the numerical range of $[A]$ in the Calkin algebra. Note that $W_{\text{ess}}(A)$ is a non-empty compact subset of complex numbers.

In the case of Calkin algebra, (2) reads as

$$\text{conv}(\sigma([A])) = \bigcap_{[S] \in \text{inv}(\mathcal{C}(\mathcal{H}))} V([S][A][S]^{-1}),$$

that is,

$$\text{conv}(\sigma_{\text{ess}}(A)) = \bigcap_{S \in \Phi(\mathcal{H})} V([S][A][S]^{-1}).$$

Since,

$$\bigcap_{S \in \Phi(\mathcal{H})} V([S][A][S]^{-1}) \subseteq \bigcap_{S \in \text{inv}(\mathcal{B}(\mathcal{H}))} V([S][A][S]^{-1}) = \bigcap_{S \in \text{inv}(\mathcal{B}(\mathcal{H}))} W_{\text{ess}}(SAS^{-1}),$$

we have

$$\text{conv}(\sigma_{\text{ess}}(A)) \subseteq \bigcap_{S \in \text{inv}(\mathcal{B}(\mathcal{H}))} W_{\text{ess}}(SAS^{-1}).$$

(3)

The goal of this paper is to show that (3) is actually an equality, see Theorem 4.

We need the notion of the Weyl spectrum. Recall that by the Atkinson Theorem (see [4, Theorem 5.17]), $A \in \mathcal{B}(\mathcal{H})$ is a Fredholm operator if and only if its range $A\mathcal{H}$ is closed and the kernels $\ker(A)$ and $\ker(A^*)$ are finite dimensional. The index of $A \in \Phi(\mathcal{H})$ is then defined as $\text{ind}(A) = \dim(\ker(A)) - \dim(\ker(A^*))$. Let $\Phi_0(\mathcal{H})$ be the set of all Fredholm operators with index 0. The Weyl spectrum of $A \in \mathcal{B}(\mathcal{H})$ is

$$\sigma_w(A) = \{\lambda \in \mathbb{C}; \quad A - \lambda I \notin \Phi_0(\mathcal{H})\}.$$ 

Note that every invertible operator is a Fredholm operator with index 0, i.e., $\text{inv}(\mathcal{B}(\mathcal{H})) \subseteq \Phi_0(\mathcal{H})$. Since $\Phi_0(\mathcal{H}) \subseteq \Phi(\mathcal{H})$, we have the following inclusions

(4) $$\sigma_{\text{ess}}(A) \subseteq \sigma_w(A) \subseteq \sigma(A).$$

Schechter proved (see [1, Theorem 2.5]) that, for any operator $A \in \mathcal{B}(\mathcal{H})$,

(5) $$\sigma_w(A) = \bigcap_{K \in \mathcal{K}(\mathcal{H})} \sigma(A + K).$$

It follows from (4) and (5) that $\sigma_w(A)$ is a non-empty compact subset of $\sigma(A)$. By [9, Theorem 9] one has

(6) $$W_{\text{ess}}(A) = \bigcap_{K \in \mathcal{K}(\mathcal{H})} \overline{W(A + K)}, \quad \text{for any } A \in \mathcal{B}(\mathcal{H}).$$
Now, since \( \sigma(A + K) \subseteq W(A + K) \) for every \( K \in \mathcal{K}(\mathcal{H}) \), we have

\[
\bigcap_{K \in \mathcal{K}(\mathcal{H})} \sigma(A + K) \subseteq \bigcap_{K \in \mathcal{K}(\mathcal{H})} W(A + K).
\]

However, by (5), the left hand side of (7) is \( \sigma_w(A) \), and by (6), the right hand side of (7) is \( \mathcal{W}_{\text{ess}}(T) \). Hence, by the convexity of the essential spectrum, \( \text{conv}(\sigma_w(A)) \subseteq \mathcal{W}_{\text{ess}}(A) \). Since the convex hulls of the essential spectrum and the Weyl spectrum coincide we conclude that

\[
\text{conv}(\sigma_{\text{ess}}(A)) = \text{conv}(\sigma_w(A)) \subseteq \mathcal{W}_{\text{ess}}(A).
\]

Now we able to prove the remained part of our main result.

**Theorem 4.** For every \( A \in \mathcal{B}(\mathcal{H}) \),

\[
\text{conv}(\sigma_{\text{ess}}(A)) = \bigcap_{S \in \text{inv}(\mathcal{B}(\mathcal{H}))} \mathcal{W}_{\text{ess}}(SAS^{-1}).
\]

**Proof.** We have to prove the inclusion

\[
\text{conv}(\sigma_{\text{ess}}(A)) \supseteq \bigcap_{S \in \text{inv}(\mathcal{B}(\mathcal{H}))} \mathcal{W}_{\text{ess}}(SAS^{-1}).
\]

By [9], there exists \( K_0 \in \mathcal{K}(\mathcal{H}) \) such that \( \sigma_w(A) = \sigma(A + K_0) \). Therefore, by (8),

\[
\text{conv}(\sigma_{\text{ess}}(A)) = \text{conv}(\sigma(A + K_0)).
\]

By the Hildebrandt’s Theorem (Theorem 1), we have

\[
\text{conv}(\sigma(A + K_0)) = \bigcap_{S \in \text{inv}(\mathcal{B}(\mathcal{H}))} W(S(A + K_0)S^{-1}),
\]

that is,

\[
\text{conv}(\sigma(A + K_0)) = \bigcap_{S \in \text{inv}(\mathcal{B}(\mathcal{H}))} W(SAS^{-1} + K_s),
\]

where \( K_s = SK_0S^{-1} \). It follows that

\[
\text{conv}(\sigma(A + K_0)) \supseteq \bigcap_{S \in \text{inv}(\mathcal{B}(\mathcal{H}))} \bigcap_{K' \in \mathcal{K}(\mathcal{H})} W(SAS^{-1} + K').
\]

By (6), the right-hand side of (10) is \( \bigcap_{S \in \text{inv}(\mathcal{B}(\mathcal{H}))} \mathcal{W}_{\text{ess}}(SAS^{-1}) \) and therefore, because of (9), we may conclude that

\[
\text{conv}(\sigma_{\text{ess}}(A)) \supseteq \bigcap_{S \in \text{inv}(\mathcal{B}(\mathcal{H}))} \mathcal{W}_{\text{ess}}(SAS^{-1}). \quad \square
\]

We conclude the paper with the following corollary of Theorem 4.

**Corollary 5.** Let \( A \in \mathcal{B}(\mathcal{H}) \). Then \( 0 \notin \text{conv}(\sigma_{\text{ess}}(A)) \) if and only if there exists a positive definite operator \( P \in \mathcal{B}(\mathcal{H}) \) such that \( 0 \notin \mathcal{W}_{\text{ess}}(PA) \).
Proof. First we will show that, for any invertible $S \in \mathcal{B}(\mathcal{H})$, zero is in $W_{ess}(SAS^{-1})$ if and only if zero is in $W_{ess}(S^*SA)$. Let $S \in \text{inv}(\mathcal{B}(\mathcal{H}))$ be arbitrary and assume that $0 \in W_{ess}(SAS^{-1})$. By (6), $0 \in W(SAS^{-1} + K)$ for every operator $K \in \mathcal{K}(\mathcal{H})$. Let $K$ be fixed. Then there exists a sequence $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ of unit vectors such that
\begin{equation}
(SAS^{-1} + K)x_n, x_n \rightarrow 0.
\end{equation}
Denote $y_n = \|S^{-1}x_n\|^{-1}S^{-1}x_n \ (n \in \mathbb{N})$. Since $1 = \|x_n\| \leq \|S\|\|S^{-1}x_n\|$ one has $\|S^{-1}x_n\|^{-2} \leq \|S\|^2$. Thus, because of (11), the sequence
\begin{equation}
\langle (S^*SA + S^*KS)y_n, y_n \rangle = \|S^{-1}x_n\|^{-2}\langle (SAS^{-1} + K)x_n, x_n \rangle
\end{equation}
converges to 0, i.e., $0 \in W(S^*SA + S^*KS)$. Since $K$ is an arbitrary compact operator and $S$ is invertible we may conclude that $0 \in W_{ess}(S^*SA)$.

For the opposite implication assume that $0 \in W_{ess}(S^*SA)$, i.e., $0 \in W(S^*SA + K)$ for every $K \in \mathcal{K}(\mathcal{H})$. Let $K$ be fixed and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ a sequence of unit vectors such that
\begin{equation}
(S^*SA + K)x_n, x_n \rightarrow 0.
\end{equation}
We denote $y_n = \|Sx_n\|^{-1}Sx_n \ (n \in \mathbb{N})$. Since $\|Sx_n\|^{-2} \leq \|S^{-1}\|^2$ for any $n$ we have
\begin{equation}
\langle (SAS^{-1} + (S^*)^{-1}KS^{-1})y_n, y_n \rangle = \|Sx_n\|^{-2}\langle (S^*SA + K)x_n, x_n \rangle \rightarrow 0.
\end{equation}
As before we conclude that $0 \in W_{ess}(SAS^{-1})$.

To finish the proof assume that $0 \notin \text{conv}(\sigma_{ess}(A))$. Then, by Theorem 4, there exists $S \in \text{inv}(\mathcal{B}(\mathcal{H}))$ such that $0 \notin W_{ess}(SAS^{-1})$, which gives $0 \notin W_{ess}(PA)$ for the positive definite operator $P = S^*S$. On the other hand, if $0 \notin W_{ess}(PA)$ for a positive definite operator $P$, then $0 \notin W_{ess}(SAS^{-1})$, where $S$ is an arbitrary invertible operator such that $P = S^*S$. By Theorem 4, $0 \notin \text{conv}(\sigma_{ess}(A))$.

Acknowledgement: The authors are grateful to Professor Vladimir Müller for his helpful discussions and suggestions.

References


Janko Bračič  
University of Ljubljana, IMFM  
Jadranska ul. 19  
SI-1000 Ljubljana  
Slovenia  
janko.bracic@fmf.uni-lj.si

Cristina Diogo  
Instituto Universitário de Lisboa  
Departamento de Matemática  
Av. das Forças Armadas  
1649-026 Lisboa, Portugal  
and  
Center for Mathematical Analysis, Geometry, and Dynamical Systems  
Mathematics Department,  
Instituto Superior Técnico, Universidade de Lisboa  
Av. Rovisco Pais, 1049-001 Lisboa, Portugal  
cristina.diogo@iscte.pt