THE PRIME SPECTRUM OF A MODULE

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Abstract. We study prime submodules in the realm of modules over not necessarily commutative rings.

1. INTRODUCTION

Prime submodules were introduced by J. Dauns in [2, 3]. Various authors have studied related questions however the most of work was done in the realm of modules over commutative rings, see, for instance, [4, 6, 7, 8, 9, 10, 11, 12], and the references cited therein. In our research we are relying on the known facts about prime submodules in commutative case and about prime ideals in (not necessarily commutative) rings. In section 2 we give a characterisation of prime submodules and show, for instance, that every maximal submodule is prime. In Section 3 we study three topologies on the spectrum of a module which generalize the Zariski topology.

Throughout the paper the letter \( R \) always stands for a ring with identity. All modules are unital. The set of all prime ideals in \( R \) is denoted by \( \text{Spec}(R) \) and called the prime spectrum of \( R \). The Zariski topology on \( \text{Spec}(R) \) is given by the family of sets \( \omega(a) = \text{Spec}(R) \setminus h(a) \), where \( h(a) = \{ p \in \text{Spec}(R); \ a \subseteq p \} \) is the hull of an ideal \( a \subseteq R \).

2. PRIME SUBMODULES

Let \( R \) be a ring and \( M \) be a left \( R \)-module. The quotient of a submodule \( N \subseteq M \) is defined by \( (N : M) = \text{ann}(M/N) \). A submodule \( N \subseteq M \) is a co-ideal if there exists a left ideal \( a \subseteq R \) such that \( N = a \cdot M = \{ \sum_{i=1}^{n} a_i \cdot x_i; \ a_i \in a, x_i \in M, i = 1, \ldots, n, n \in \mathbb{N} \} \). For a left ideal \( a \subseteq R \), we shall say that \( a \cdot M \) is its co-ideal. A module \( M \) is a multiplication module (see [5]) if each submodule in \( M \) is a co-ideal.

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The following proposition lists several useful properties of quotients.
An easy proof is omitted.

**Proposition 2.1.** Let \( R \) be a ring and \( M \) a left \( R \)-module.

(i) For submodules \( N \subseteq L \subseteq M \), we have \( (N : M) \subseteq (L : M) \).

(ii) If \( \{N_i\}_{i \in I} \) is an arbitrary family of submodules in \( M \), then

\[
(\bigcap_{i \in I} N_i : M) = \bigcap_{i \in I} (N_i : M).
\]

Let \( R \) be a ring and \( M \) be a left \( R \)-module. In [2] Dauns has defined prime submodules as follows. A proper submodule \( P \subset M \) is prime if, for \( a \in R \) and \( x \in M \), the inclusion \( (aR) \cdot x \subseteq P \) implies \( a \in (P : M) \) or \( x \in P \).

The set of all prime submodules in a left \( R \)-module \( M \) will be denoted by \( \text{Spec}(M) \) and called the prime spectrum of \( M \). A module \( M \) is said to be primeless ([11]) if \( \text{Spec}(M) \) is empty. From now on we shall always assume that a module under consideration is not primeless.

**Theorem 2.2.** Let \( R \) be a ring and \( M \) be a left \( R \)-module. For a proper submodule \( P \) in \( M \), the following assertions are equivalent.

(a) For a two-sided ideal \( a \subseteq R \) and a submodule \( N \subseteq M \), the inclusion \( a \cdot N \subseteq P \) implies \( a \subseteq (P : M) \) or \( N \subseteq P \).

(b) \( P \) is prime.

(c) For a left (or right) ideal \( a \subseteq R \) and a submodule \( N \subseteq M \), the inclusion \( a \cdot N \subseteq P \) implies \( a \subseteq (P : M) \) or \( N \subseteq P \).

**Proof.** (a) \( \Rightarrow \) (b) Let \( a \in R \) and \( x \in M \) be such that \( a \cdot R \cdot x \subseteq P \). It follows, since \( P \) is a submodule, that \( (RaR) \cdot (R \cdot x) \subseteq P \). Now, by (a), we get \( RaR \subseteq (P : M) \) or \( R \cdot x \subseteq P \). If the former is true, then \( a \in (P : M) \), and the latter inclusion gives \( x \in P \).

(b) \( \Rightarrow \) (c) Assume that \( a \cdot N \subseteq P \), for a left (right) ideal \( a \subseteq R \) and a submodule \( N \subseteq M \). If \( N \nsubseteq P \), then there exists \( y \in N \setminus P \). For every \( a \in a \), we have \( (aR) \cdot y = a \cdot (R \cdot y) \subseteq a \cdot N \subseteq P \) which gives, by (b), \( a \in (P : M) \).

(c) \( \Rightarrow \) (a) Obvious. \( \square \)

**Corollary 2.3.** If \( R \) is a commutative ring and \( M \) is a left \( R \)-module, then a proper submodule \( P \subset M \) is a prime submodule if and only if, for arbitrary \( a \in R \) and \( x \in M \), we have \( a \in (P : M) \) or \( x \in P \) whenever \( a \cdot x \in P \).
Proof. This is clear by the equivalence

\[ a \cdot x \in P \quad \text{if and only if} \quad Ra \cdot x \subseteq P. \]

Consider \( R \) as a left \( R \)-module over itself. It is straightforward that \( a = (a : R) \) if \( a \subseteq R \) is a two-sided ideal. Thus, if we replace in Theorem 2.2 and Corollary 2.3 the phrase \( a \) proper submodule \( P \) with the phrase \( a \) proper two-sided ideal \( p \), we get two well-known assertions. It follows, for instance, that every prime ideal in \( R \) is a prime submodule when \( R \) is considered as a left module over itself. If \( R \) is commutative all submodules are of this type.

In the following theorem, which should be compared with Lemma 1.1 in [11], we use a simple fact that \( \frac{M}{N} \) is a left \( \frac{R}{(N : M)} \)-module with multiplication \((a + (N : M)) \cdot (x + N) = a \cdot x + N\), where \( a + (N : M) \in \frac{R}{(N : M)} \) and \( x + N \in M/N \) are arbitrary, whenever \( N \) is a submodule in a left \( R \)-module \( M \).

Note that any \( R \)-module \( X \) is an \( \frac{R}{\text{ann}(X)} \)-module as well.

**Theorem 2.4.** Let \( R \) be a ring and \( M \) be a left \( R \)-module. A proper submodule \( P \subseteq M \) is prime if and only if the quotient \( (P : M) \) is a prime ideal in \( R \) and \( (0) \) is a prime submodule in the \( \frac{R}{(P : M)} \)-module \( M/P \).

Proof. Let \( P \) be a prime submodule in \( M \). Since \( P \) is proper, the identity of \( R \) is not in the quotient of \( P \), which means that \( (P : M) \) is a proper ideal in \( R \). Let \( a \) and \( b \) be two-sided ideals in \( R \) such that \( ab \subseteq (P : M) \). Assume that \( b \not\subseteq (P : M) \). Choose \( b \in b \) which is not in \( (P : M) \). Then there exists \( x \in M \) such that \( y = b \cdot x \) is not in \( P \). It is clear that \( x \) is not in \( P \). Let \( a \in a \) be arbitrary. It follows from the inclusion \( ab \subseteq (P : M) \) that \( aRb \subseteq (P : M) \) and consequently \((aR) \cdot y = (aRb) \cdot x \subseteq P\), which gives \((RaR) \cdot (R \cdot y) \subseteq P\). By the assumption, \( P \) is a prime submodule, which means that we have \( RaR \subseteq (P : M) \) or \( R \cdot y \subseteq P \). However, the latter case is impossible because of \( y \notin P \). Hence \( a \in RaR \subseteq (P : M) \), which gives \( a \subseteq (P : M) \).

Assume that \( a + (P : M) \in \frac{R}{(P : M)} \) and \( x + P \in M/P \) are such that \((a + (P : M))R/(P : M) \cdot (x + P) \subseteq (0) \). Then we have \( aR \cdot x \subseteq P \) and consequently \( a \in (P : M) \) or \( x \in P \), by Theorem 2.2. Now we conclude that \( a + (P : M) \in ((0) : M/P) \) or \( x + P \in (0) \), which means, by Theorem 2.2, that \( (0) \) is a prime submodule in a left \( \frac{R}{(P : M)} \)-module \( M/P \). (Note that \((0) : M/P\) is the quotient of the
trivial submodule in the left $R/(P : M)$-module $M/P$ and that it is a trivial ideal in $R/(P : M)$.

In order to prove the opposite implication assume that a two-sided ideal $a \subseteq R$ and a submodule $N \subseteq M$ are such that $a \cdot N \subseteq P$. It is easily seen that $a(N : M) \subseteq (P : M)$. Since, by the assumption, $(P : M)$ is a prime ideal, we conclude that $a \subseteq (P : M)$ or $(N : M) \subseteq (P : M)$. If the former is true, we have done. Assume therefore that $a \not\subseteq (P : M)$ and let $a \in a \setminus (P : M)$. Then, for an arbitrary $y \in N$, the inclusion $(a+(P : M))R/(P : M) \cdot (y+P) \subseteq (0)$ gives $a+(P : M) \in ((0) : M/P)$ or $y+P \in (0)$, by the assumption. However, the former is not the case because of $a \not\subseteq (P : M)$. Thus, $y \in P$.

Corollary 2.5. Let $R$ be a ring and $M$ be a left $R$-module. There is a well defined mapping $\nu : \text{Spec}(M) \to \text{Spec}(R)$ which is given by

$$\nu(P) = (P : M).$$

Following [4] we call $\nu$ the natural mapping. For a prime ideal $p \in \text{Spec}(R)$, let $\text{Spec}_p(M) = \nu^{-1}({\{p\}})$. By Proposition 2.1 (i), it is clear that $\text{Spec}_p(M)$ is empty if $\text{ann}(M) \not\subseteq p$. Also, if $p \cdot M = M$, for $p \in \text{Spec}(R)$, then $\text{Spec}_p(M) = \emptyset$. On the other hand, if $m \in \text{Spec}(R)$ is maximal, then $\text{Spec}_m(M)$ is non-empty if and only if $m \cdot M \neq M$.

In the rest of this section we study maximal submodules.

Definition 2.6. A submodule $N$ in a left $R$-module $M$ is co-cyclic if $M/N$ is a cyclic $R$-module. An element $u \in M$ is co-cyclic for $N$ if $u + N$ is cyclic for $M/N$.

It is obvious that a proper submodule does not contain co-cyclic elements. In the following proposition we list some properties of co-cyclic and maximal submodules.

Proposition 2.7. Let $R$ be a ring and $M$ a left $R$-module.

(i) If $L \subseteq M$ is a submodule such that it contains a co-cyclic submodule $N \subseteq M$, then $L$ is co-cyclic as well. Moreover, if $u \in M$ is co-cyclic for $N$, then it is co-cyclic for $L$.

(ii) If $K \subseteq X$ is a maximal submodule, then $M/K$ is simple. Thus, $K$ is co-cyclic.

(iii) If $K$ is a maximal submodule in $M$ and $x \in M$ is not in $K$, then $R \cdot x + K = M$.

(iv) Every proper co-cyclic submodule is included in a maximal submodule.
(v) Every maximal submodule in $M$ is prime.

Proof. The claims (i), (ii), and (iii) are easily proven.

(iv) Let $N$ be a proper co-cyclic submodule with a co-cyclic element $u$. Denote by $\mathcal{F}$ the family of all the submodules in $M$ that contain $N$ and do not contain $u$. This family is partially ordered. The union of all submodules in a given linearly ordered subfamily of $\mathcal{F}$ is the upper bound of this subfamily. Thus, by Zorn’s lemma, the family $\mathcal{F}$ has the maximal element, say $K$. If $K$ were not a maximal submodule, there would be a maximal submodule $K'$ such that $N \subseteq K \subset K'$ and $u \in K'$. However, by (i), this is impossible since $u$ has to be co-cyclic for $K'$.

(v) Let $K$ be a maximal submodule. Assume that $a \in R$ and $x \in M$ are such that $aR \cdot x \subseteq K$. Suppose that $x \notin K$. Then $x + K$ is a non-zero element in $M/K$, which means that $x + K$ is cyclic for this module. Hence, for every $y \in M$, there exists $b \in R$ such that $y + K = b \cdot (x + K)$. It follows that $y - b \cdot x \in K$ and therefore $a \cdot y - ab \cdot x \in K$. However, by the assumption, $ab \cdot x \in K$ and we conclude $a \cdot y \in K$ and consequently $a \in (K : M)$. □

It follows, by the statement (v) of the previous proposition and by Theorem 2.4, that the quotient of a maximal submodule is a prime ideal. However, it is not necessarily that the quotient of a maximal submodule is a maximal ideal.

3. Topologies on the prime spectrum

Let $R$ be a ring and $M$ be a left $R$-module. For a given non-empty subset $S \subseteq \text{Spec}(M)$, the intersection of all submodules in $S$ is denoted by $k(S)$ and we set $k(\emptyset) = M$.

**Definition 3.1.** The outer hull of a submodule $N \subseteq M$ is

$$v(N) = \{ P \in \text{Spec}(M); \ (N : M) \subseteq (P : M) \}$$

and the inner hull of a submodule $N \subseteq M$ is

$$h(N) = \{ P \in \text{Spec}(M); \ N \subseteq P \}.$$ 

The assertions in the following lemma are straightforward to prove.

**Lemma 3.2.** (i) For every submodule $N \subseteq M$, we have $h(N) \subseteq v(N)$.

(ii) $v(0) = h(0) = \text{Spec}(M)$ and $v(M) = h(M) = \emptyset$.

(iii) If $S_1 \subseteq S_2$, then $k(S_1) \supseteq k(S_2)$. 

(iv) $S \subseteq h(k(S)) \subseteq v(k(S))$, for every $S \subseteq \text{Spec}(M)$.

(v) If $N \subseteq L \subseteq M$, then $v(N) \supseteq v(L)$ and $h(N) \supseteq h(L)$.

(vi) For $S \subseteq \text{Spec}(M)$, we have $k(v(k(S))) \subseteq k(h(k(S))) = k(S)$; in general the inclusion might be proper.

(vii) Equalities $v(N) = v(k(v(N)))$ and $h(N) = h(k(h(N)))$ hold for any submodule $N \subseteq M$.

(viii) For an arbitrary family $\{N_i\}_{i \in I}$ of submodules in $M$, we have

$$\bigcap_{i \in I} h(N_i) = h\left(\sum_{i \in I} N_i\right).$$

(ix) If $N$ and $L$ are submodules in $M$, then $h(N) \cup h(L) \subseteq h(N \cap L)$.

Let us show that the family of outer hulls is closed under arbitrary intersections and finite unions.

**Proposition 3.3.** Let $R$ be a ring and $M$ a left $R$-module.

(i) For an arbitrary family $\{N_i\}_{i \in I}$ of submodules in $M$, we have

$$\bigcap_{i \in I} v(N_i) = v\left(\sum_{i \in I} (N_i : M) \cdot M\right).$$

(ii) If $N$ and $L$ are submodules in $M$, then $v(N) \cup v(L) = v(N \cap L)$.

*Proof.* (i) If $P \in \cap_{i \in I} v(N_i)$, then $(N_i : M) \subseteq (P : M)$, for all $i \in I$. It follows $(N_i : M) \cdot M \subseteq (P : M) \cdot M$, for all $i \in I$, and consequently

$$\sum_{i \in I} (N_i : M) \cdot M \subseteq (P : M) \cdot M.$$

Since $(P : M) \cdot M \subseteq P$, we conclude

$$\left(\sum_{i \in I} (N_i : M) \cdot M : M\right) \subseteq (P : M).$$

On the other hand, if $P$ is in the outer hull of $\sum_{i \in I} (N_i : M) \cdot M$, then

$$(N_j : M) \subseteq ((N_j : M) : M : M) \subseteq (\sum_{i \in I} (N_i : M) : M : M) \subseteq (P : M)$$

gives $P \in v(N_j)$, for all $j \in I$.

(ii) It is easily seen that $(N : M)(L : M) \subseteq (N \cap L : M)$, for any pair of submodules $N$, $L \subseteq M$. Thus, if $P \in v(N \cap L)$, then

$$(N : M)(L : M) \subseteq (N \cap L : M) \subseteq (P : M)$$

gives $(N : M) \subseteq (P : M)$ or $(N : M) \subseteq (P : M)$, by primeness of $(P : M)$ (see Theorem 2.4). The opposite inclusion is easily seen.  \(\square\)
Let $R$ be a ring and $M$ be a left $R$-module. For each submodule $N \subseteq M$, let us denote $\omega(N) = \text{Spec}(M) \setminus \mathcal{v}(N)$.

**Theorem 3.4** (cf. [4]). The family

$$\tau = \{ \omega(N); \ N \text{ is a submodule in } M \}$$

is a topology on $\text{Spec}(M)$.

**Proof.** Since $\omega((0)) = \emptyset$ and $\omega(M) = \text{Spec}(M)$ we have $\emptyset \in \tau$ and $\text{Spec}(M) \in \tau$. Using the claim (i) of Proposition 3.3 we get

$$\bigcup_{i \in I} \omega(N_i) = \omega\left( \sum_{i \in I} (N_i : M) \cdot M \right),$$

for an arbitrary family $\{N_i\}_{i \in I}$ of submodules in $M$. This shows that $\tau$ is closed under arbitrary unions. Similarly, by the statement (ii) of Proposition 3.3, we have $\omega(N) \cap \omega(L) = \omega(N \cap L)$, for all submodules $N$ and $L$ in $M$, which shows that $\tau$ is closed under finite intersections. □

Let $\tau' = \{ \omega(a \cdot M); \ a \text{ is a left ideal in } R \}$. If $R$ is a commutative ring and $M$ is a left module over it, then $\tau'$ is a topology on $\text{Spec}(M)$ (see [8]). We are going to prove that this is true in general.

**Lemma 3.5** (cf. Lemma 3.1 in [11]). Let $R$ be a ring and $M$ be a left $R$-module. For a left ideal $a \subseteq R$ and a submodule $N \subseteq M$, we have

$$\mathcal{v}(a \cdot M) \cup \mathcal{v}(N) = \mathcal{v}(a \cdot N) = \mathcal{v}(a \cdot M \cap N).$$

**Proof.** We shall follow the proof of Lemma 3.1 in [11] modifying the arguments when needed. The inclusions

$$\mathcal{v}(a \cdot M) \cup \mathcal{v}(N) \subseteq \mathcal{v}(a \cdot M \cap N) \subseteq \mathcal{v}(a \cdot N)$$

are obvious. Suppose that $P \in \mathcal{v}(a \cdot N)$. Then $(a \cdot N : M) \subseteq (P : M)$. It is easily seen that $a(N : M) \subseteq (a \cdot N : M)$. Since, by Theorem 2.4, the quotient $(P : M)$ is a prime ideal the inclusion $a(N : M) \subseteq (P : M)$ gives $a \subseteq (P : M)$ or $(N : M) \subseteq (P : M)$. If the former is true, then we have the inclusion $a \cdot M \subseteq P$ and consequently $(a \cdot M : M) \subseteq (P : M)$, which proves the assertion. Obviously, if $(N : M) \subseteq (P : M)$ we have done as well. □

**Corollary 3.6.** For arbitrary left ideals $a$ and $b$ in $R$, we have

$$\mathcal{v}(a \cdot M) \cup \mathcal{v}(b \cdot M) = \mathcal{v}(ab \cdot M) = \mathcal{v}(a \cdot M \cap b \cdot M).$$
Theorem 3.7. Let $R$ be a ring and $M$ be a left $R$-module. The family $\tau'$ is a topology on Spec($M$).

Proof. Use the claim (i) of Proposition 3.3 and Corollary 3.6. \hfill \square

It is obvious that $\tau' \subseteq \tau$. Note however that the topology $\tau$ is also quite weak. For instance, the $\tau$-closure of a singleton $\{P\} \subseteq \text{Spec}(M)$ is $\text{Spec}_{(P:M)}(M)$, a set that might be quite large (see [1]).

For a submodule $N$ in a left $R$-module $M$, set $\omega^*(N) = \text{Spec}(M) \setminus h(N)$ and let $\tau^* = \{\omega^*(N); \ N \text{ is a submodule in } M\}$. It is well-known that $\tau^*$ is not always a topology on Spec($M$). If it is, the module is said to be a top module (see [11]).

Theorem 3.8. Every multiplication module is a top module.

Proof. The proof of Lemma 3.1 in [11] works also in the case when the involved ring is not commutative. Thus, we may use Corollary 3.2 in [11], which says $h(a \cdot M) \cup h(b \cdot M) = h(ab \cdot M)$. Since the module under consideration is a multiplication module the last equality shows that $\tau^*$ is closed under finite intersections. That $\tau^*$ is closed under arbitrary unions follows from Lemma 3.2(viii). \hfill \square

At the end let us show that the natural mapping is continuous.

Theorem 3.9. Let $R$ be a ring and $M$ be a left $R$-module. The natural mapping $\nu : \text{Spec}(M) \rightarrow \text{Spec}(R)$ is continuous if Spec($M$) is endowed with the topology $\tau'$ and Spec($R$) has the Zariski topology. More precisely, for every left ideal $a \subseteq R$, we have $\nu^{-1}(h(a)) = v(a \cdot M)$.

Proof. It is enough to prove the last equality. If $P \in \text{Spec}(M)$ is in $\nu^{-1}(h(a))$, then $a \subseteq (P : M)$ and consequently $a \cdot M \subseteq (P : M) : M \subseteq P$, which gives $(a \cdot M : M) \subseteq (P : M)$. On the other hand, if $P \in \text{Spec}(M)$ is in $v(a \cdot M)$, then $a \subseteq (a \cdot M : M) \subseteq (P : M)$ and therefore $(P : M) \subseteq \nu^{-1}(h(a))$. \hfill \square

Corollary 3.10 (cf. [4]). The natural mapping is continuous if Spec($M$) is endowed with the topology $\tau$.

Corollary 3.11. Let $M$ be a multiplication module. Then the natural mapping is continuous if Spec($M$) is endowed with the topology $\tau^*$.

Proof. Since $h(a \cdot M) \subseteq v(a \cdot M)$ we have $h(a \cdot M) \subseteq \nu^{-1}(h(a))$, by Theorem 3.9. The opposite inclusion is proved in the first part of the proof of Theorem 3.9. Hence $\nu^{-1}(h(a)) = h(a \cdot M)$. \hfill \square
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References


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